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A mathematical model and analytical solution for buckling of inclined beam-columns

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Abstract

The differential equation that governs the buckling behavior of an inclined beam-column is obtained using the energy method, and the use of a suitable change of variable reduces the various geometrical and physical parameters into a single dimensionless length parameter (X). An exact solution is presented via the use of some members of the family of generalized hypergeometric functions. One type of boundary condition (pinned-ends) is presented and analyzed, and others boundary conditions may be easily studied following the same process. The analysis shows the singular behavior of the inclined beam-column and demonstrates the procedure used to obtain the critical values of the axial force. An important application of this model is its usefulness in the analysis of buckling of drillstrings within curved boreholes. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

Mechanical buckling is the imminent or instantaneous collapse of a structure due to external and/or internal loads which, however, would not be sufficiently intense to cause mechanical yield of the material in the structure. The phenomenon of mechanical buckling of columns and beams has been studied extensively, with various models attempting to account for the myriad physical conditions that determine the behavior of the system. Several classical problems have been treated in the existing literature such as the Euler's column, the flagpole model [1–3], and beam-column model [2].

We will be interested in the buckling behavior of *inclined beam-columns*. This phenomenon has not been satisfactorily addressed in the literature, but is of central importance, for example, in the analysis of mechanical buckling of drillstrings within curved boreholes. The problems arising from this phenomenon have been a major topic of interest in the petroleum industry for more than 40 years.

Some of the factors considered in a reasonable model might be the distributed weight of the inclined beam-column, whether the ends are pinned, built-in, or free, the inclination of the col-

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umn, and whether a driving load is present. Reinforcement or stiffening of the column by the presence of rings or collars has also been recently studied [4]. The most common driving loads treated are axial and lateral forces, moments, torque, and pressure.

A fruitful approach has been to analyze the total potential energy of the system and to investigate conditions that lead to states of stable or unstable equilibrium of the system. Abrupt changes in these states may be mathematically characterized by bifurcations, eigenvalues, or singularities.

The study that we present here may be termed an *inclined beam-column theory*, incorporating both a distributed weight and the inclination of the beam-column. We use the energy method to derive a differential equation and associated boundary conditions. A closed-form solution for this enhanced model is demonstrated via the use of generalized hypergeometric functions, which allows a careful analysis of the critical forces that cause buckling. In addition to being a useful model in its own right, it provides a basis for a fully analytical model for buckling behavior under curved constraints (as in the case of the buckling of drillstrings within curved boreholes). In this paper we limit our discussion to demonstration of the inclined beam-column theory, deferring a study of its application to the curved borehole problem to a future presentation.

2. Preliminaries: Models featuring distributed weight

Before presenting the inclined beam-column theory, which is the main contribution of this paper, we briefly discuss two familiar models, their solutions and implications. Both models consider the distributed weight w of the column, but neither incorporates the inclination as a parameter. We choose these two models as illustrations of the eigenvalue case and the singularity case.

The *flagpole problem* is a classical problem in elasticity and advanced mechanics and has been discussed broadly by several investigators [1–3]. The problem consists of finding the equilibrium configuration of a column vertically built-in at the lower end, with the upper end free, loaded by the distributed weight along its length. The differential equation and boundary conditions for this problem are given by

$$EI\frac{d^3y}{dx^3} + w(l-x)\frac{dy}{dx} = 0,$$
$$y(0) = y'(0) = y''(l) = 0,$$

where x is the position along the pole of length l, $y(x)$ is the lateral displacement, and EI is the stiffness [1] of the pole. The trivial function $y(x) \equiv 0$ solves the differential equation and satisfies the boundary conditions. Non-trivial solutions exist for particular discrete values of pole length l, thereby defining this as an eigenvalue problem. An exact solution was obtained by Love [1]:

$$p(x) = \frac{dy(x)}{dx} = \left[aJ_{1/3}(\xi) + bJ_{-1/3}(\xi)\right](1-x)^{1/2},$$
$$\xi(x) = \frac{2}{3}\sqrt{\frac{w}{EI}}(1-x)^{3/2},$$

where a and b are constants to be determined. (This expression has to be integrated once to obtain $y(x)$.) The first critical length that causes instability of the column, here expressed as l_{crit}, is given by

[1] E is the Young's modulus of the material and I is the moment of inertia of the beam-column cross-section.

$$l_{\text{crit}} = 1.986\sqrt[3]{\frac{EI}{w}}.$$

Another related problem is the case of a vertical pole with pinned-ends. Although this does not represent a flagpole, the nomenclature is kept because it involves essentially the same differential equation with different boundary conditions. The problem is described by the following equations:

$$EI\frac{d^4y}{dx^4} + \frac{d}{dx}\left(w(l-x)\frac{dy}{dx}\right) = 0,$$

$$y(0) = y(l) = y''(0) = y''(l) = 0.$$

A *beam-column* is a beam subjected to an axial compressive force in addition to a lateral force. The solution and analysis of this problem are also in the literature [2]. The differential equation is given by

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = q(x) \not\equiv 0,$$

where P is the axial force and $q(x)$ the distributed weight. Now the trivial function $y(x) \equiv 0$ is not a solution of the differential equation. For $q(x) = q_0$, a solution for the differential equation is

$$y(x) = \frac{q_0 l^4}{16EIu^4}\left[\frac{\cos(u - 2ux/l)}{\cos(u)} - 1\right] - \frac{q_0 l^2}{8EIu^2}x(l-x),$$

$$u = \frac{1}{2}\sqrt{\frac{P}{EI}}.$$

No solution exists for $u = (k + 1/2)\pi$, for k a natural number. Therefore, the value $u = \pi/2$ represents the first critical value of the system. From the definition of u we find

$$\frac{\pi}{2} = \frac{l}{2}\sqrt{\frac{P_{\text{crit}}}{EI}} \rightarrow P_{\text{crit}} = \frac{\pi^2 EI}{l^2}.$$

Although this expression is the same as the critical force for an Euler's column, the behavior is significantly different. For the Euler's column it represents an eigenvalue of the problem and for the beam-column it represents a *singularity*.

For the beam-column, there is always a non-trivial solution except for special discrete values of the axial compressive force in which no solution exists at all; the smallest such force is identified as P_{crit} above.

3. The inclined beam-column theory

In this section we present a model that incorporates both the distributed weight w and the inclination θ of the beam-column. Fig. 1 presents a schematic of a loaded inclined beam-column before buckling (a), and after buckling (b). The derivation of the differential equation that describes this model was accomplished via the energy method and is presented in Appendix A. The energy method is based upon the principle of virtual work, which states that if an elastic body or system in equilibrium is given a small displacement or deformation, then the work done by all external and body forces acting on the system equals the increase of strain energy stored in the system. Using the concept of potential energy, we say that if a system in equilibrium is given a small displacement, the total potential energy does not change, that is, the total potential energy is stationary. The expression for the total potential energy was found as

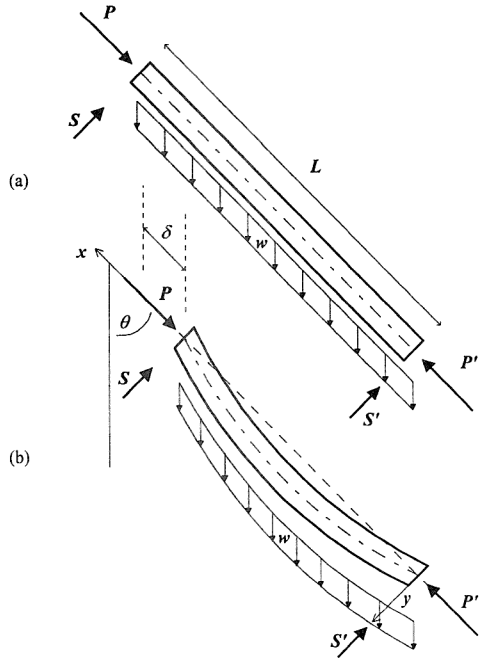

Fig. 1. A weighted inclined beam-column under axial force.

$$U(y) = \int_0^L \left\{ \frac{EI}{2}(y'')^2 - \frac{1}{2}[P + w_c(L-x)](y')^2 - w_s y \right\} \mathrm{d}x, \tag{1}$$

where the parameters are identified as follows: w is the distributed weight (weight/length), $w_c = w\cos\theta$, $w_s = w\sin\theta$, θ is the angle of inclination of the beam-column from the vertical.

The function $y(x)$ represents any admissible configuration that the inclined beam-column may assume. The equilibrium configuration is that in which the total potential energy $U(y)$ becomes stationary. Applying a standard procedure from the Calculus of Variations we arrive at the following differential equation and corresponding boundary conditions:

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}x^2}[EIy''] + \frac{\mathrm{d}}{\mathrm{d}x}\{[P + w_c(L-x)]y'\} - w_s = 0, \\ &y(0) = y(L) = y''(0) = y''(L) = 0. \end{aligned} \tag{2}$$

4. Simplification of the differential equation

For homogeneous beam-columns with constant cross-section, the differential equation may be readily integrated once to arrive at the third order equation

$$y''' + \frac{1}{EI}[P + w_c(L-x)]y' = \frac{w_s}{EI}x + \frac{c}{EI},$$

where c is a constant to be determined. Next we make a linear change of variable involving the independent variable x in order to simplify the coefficient of the term containing y'. With a change of variable of the form

$$x = aX + b \quad \text{or} \quad X = \frac{x-b}{a}$$

(where a and b are constants to be determined) and repeated application of chain rules for differentiation, we arrive at

$$\frac{d^3y}{dX^3} + \frac{1}{EI}[P + w_c(L - aX - b)]a^2\frac{dy}{dX} = a^3\frac{w_s}{EI}(aX + b) + a^3\frac{c}{EI}.$$

A further and quite important simplification will result if the constants a and b are chosen so that the coefficient of y' becomes $-X$. This can be accomplished with the choice:

$$a = \left(\frac{EI}{w_c}\right)^{1/3},$$

$$b = \frac{P}{w_c} + L.$$

The linear transformation then becomes:

$$x = \left(\frac{EI}{w_c}\right)^{1/3} X + \left(\frac{P}{w_c} + L\right).$$

Also, X is a dimensionless variable having the form

$$X = -\left(\frac{w_c}{EI}\right)^{1/3}\left(\frac{P}{w_c} + L - x\right).$$

Applying the specified change of variable to the differential equation one finds the equivalent but more convenient form

$$y''' - Xy' = \alpha X + 2b_3,$$

where

$$\alpha = \left(\frac{EI}{w_c}\right)^{1/3}\tan(\theta),$$

$$b_3 = \frac{1}{2}\left[\left(\frac{P}{w_c} + L\right)\tan(\theta) + \frac{c}{w_c}\right].$$

The constant α depends only on the parameters of the inclined beam-column. The constant b_3 however is unspecified due to the unknown integration constant c. The same change of variable is applied to the boundary conditions. If X_1 and X_2 map the values for $x = 0$ and $x = L$ (the lower and upper ends of the column) the boundaries are now given by

$$X_1 = -\left(\frac{EI}{w_c}\right)^{-1/3}\left(\frac{P}{w_c} + L\right),$$

$$X_2 = -\left(\frac{EI}{w_c}\right)^{-1/3}\left(\frac{P}{w_c}\right), \tag{3}$$

and the length of the column in the new scale is

$$L_D = X_2 - X_1 = \left(\frac{EI}{w_c}\right)^{-1/3} L.$$

The dependent variable y is not affected by the change of variable. Therefore, since the conditions on the second derivative y'' are homogeneous, the problem, using the new dimensionless variable X, becomes:

$$\begin{aligned} &y''' - Xy' = \alpha X + 2b_3, \\ &y(X_1) = y(X_2) = y''(X_1) = y''(X_2) = 0. \end{aligned} \tag{4}$$

5. General solution of the differential equation

Since the differential equation contains no terms in $y(X)$, a reduction of order may be readily achieved by the substitution $z(X) = y'(X)$, resulting in the second order equation

$$z'' - Xz = \alpha X + 2b_3.$$

This differential equation does not have solutions expressed in terms of elementary functions. Its homogeneous form is Airy's differential equation whose solutions are written in terms of the Airy functions $\mathrm{Ai}(X)$ and $\mathrm{Bi}(X)$ as follows:

$$z(X) = c_1\mathrm{Ai}(X) + c_2\mathrm{Bi}(X).$$

The solution of the non-homogeneous form can be obtained formally using variation of parameters. The Wronskian $W(\mathrm{Ai}, \mathrm{Bi})$ is constant and equal to $1/\pi$ and the solution can be written as

$$z(X) = \left[D - \pi \int (\alpha X + 2b_3)\mathrm{Bi}\, \mathrm{d}X\right]\mathrm{Ai} + \left[E - \pi \int (\alpha X + 2b_3)\mathrm{Ai}\, \mathrm{d}X\right]\mathrm{Bi},$$

where b_3, D and E are constants to be determined. The function $z(X)$ must be integrated to determine $y(X)$. This introduces the fourth integration constant required by the initial fourth order differential equation.

Another approach is to obtain a solution by power series, taking advantage of the fact that any point $X \neq \infty$ is an ordinary point. After suitable series manipulations, the solution is expressed in closed form in terms of some members of the generalized hypergeometric functions. The general solution and its first and second derivatives are written as (see Ref. [5])

$$y(X) = b_1M_1(X) + b_2M_2(X) + b_3M_3(X) + b_4 - \alpha X,$$

$$y'(X) = b_1R_1(X) + b_2R_2(X) + b_3R_3(X) - \alpha,$$

$$y''(X) = b_1N_1(X) + b_2N_2(X) + b_3N_3(X),$$

where the functions M_n, R_n and N_n, for $n = 1,2,3$ are given by

$$M_1(X) = X \cdot {}_1F_2\left(\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; \frac{X^3}{9}\right),$$

$$M_2(X) = \frac{X^2}{2} \cdot {}_1F_2\left(\frac{2}{3}; \frac{4}{3}, \frac{5}{3}; \frac{X^3}{9}\right),$$

$$M_3(X) = \frac{X^3}{3} \cdot {}_2F_3\left(1, 1; \frac{4}{3}, \frac{5}{3}, 2; \frac{X^3}{9}\right),$$

$$R_1(X) = {}_0F_1\left(\frac{2}{3};\frac{X^3}{9}\right),$$

$$R_2(X) = X \cdot {}_0F_1\left(\frac{4}{3};\frac{X^3}{9}\right),$$

$$R_3(X) = X^2 \cdot {}_1F_2\left(1;\frac{4}{3},\frac{5}{3};\frac{X^3}{9}\right),$$

$$N_1(X) = \frac{X^2}{2} \cdot {}_0F_1\left(\frac{5}{3};\frac{X^3}{9}\right),$$

$$N_2(X) = {}_0F_1\left(\frac{1}{3};\frac{X^3}{9}\right),$$

$$N_3(X) = 2X \cdot {}_1F_2\left(1;\frac{2}{3},\frac{4}{3};\frac{X^3}{9}\right).$$

Imposing the boundary conditions for the pinned-ends configuration results in the following non-homogeneous system of equations:

$$\begin{bmatrix} M_1(X_1) & M_2(X_1) & M_3(X_1) & 1 \\ M_1(X_2) & M_2(X_2) & M_3(X_2) & 1 \\ N_1(X_1) & N_2(X_1) & N_3(X_1) & 0 \\ N_1(X_2) & N_2(X_2) & N_3(X_2) & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix} = \alpha \begin{Bmatrix} X_1 \\ X_2 \\ 0 \\ 0 \end{Bmatrix}. \tag{5}$$

This system can be written as

$$\mathbf{S}(X_1, X_2)\mathbf{b} = \alpha\mathbf{X}.$$

The length of the column relates to the dimensionless values X_1 and X_2, that is,

$$X_1 = X_2 - L_D.$$

In addition, for a given configuration, X_2 is linearly related to the compressive force P via Eq. (3). Therefore P and X_2 may be used interchangeably and the reference will be made to the most appropriate for the context, i.e., the variable P is used when dealing with force and the variable X_2 is used when dealing with the solution of the differential equation. Therefore we can write:

$$\mathbf{S}(X_2)\mathbf{b} = \mathbf{S}(P)\mathbf{b} = \alpha\mathbf{X}.$$

Singularities occur for instances of X_2 or P in which the determinant of the boundary matrix $\mathbf{S}$ vanishes.

6. Analysis of the solution for the generalized beam-column

In this section we determine the general shape of the center line of the inclined beam-column and analyze both the behavior and the change in the behavior when combinations of the parameters L, EI, w, θ, and P put the inclined beam-column close to critical values. These are the values that are associated with the phenomenon of buckling. We find the critical values of the parameters (in particular the critical axial force P_{crit}) by determining those values of P for which the determinant $\mathbf{S}(P)$ is singular and for which there is thus no solution for the displacement $y(x)$.

A first step in accomplishing this analysis is to evaluate the hypergeometric functions which comprise the general solution. Three methods were applied in this study. (i) Evaluations for

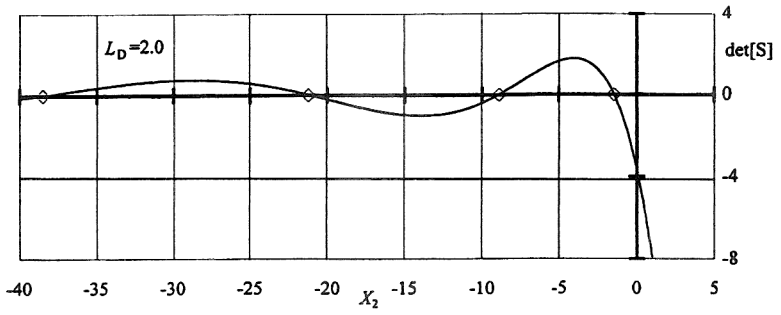

Fig. 2. Roots of the boundary determinant for $L_D = 2.0$.

smaller values of X, which are normally applicable for analyses close to and less than the first singularity, are easily performed using series expansion of the functions. Evaluations for larger values of X, which are normally applicable for higher buckling modes, series expansion becomes impractical due to the truncations occurring in the terms of large power. [2] (ii) A second approach is the direct integration of the particular differential equations for which those hypergeometric functions are solutions. This can be accomplished using a Runge–Kutta integrator with adaptive stepsize. This method is very accurate; however the time needed for calculation, in particular when thousands of calculations are required as in an iterative process, rendered this method impractical. (iii) A third method, which is suitable for large values of X, is the asymptotic expansion of the functions [5]. A final practical implementation was obtained using series expansion for small values of X and asymptotic expansions for large values of X.

The change of variable applied to the differential equation introduced the dimensionless parameter L_D which accounts for the length L, the stiffness EI, the distributed weight w, and the inclination θ of the beam-column. In addition, the value of the parameter α is defined from those physical parameters. Thus, the only parameters that need be considered for a given configuration are L_D and P (or X_2). We analyze the solution by selecting a representative inclined beam-column with fixed L_D and α.

In order to determine the critical forces, we express the boundary matrix $\mathbf{S}$ as a function of a single variable X_2, and seek the values of X_2 for which its determinant vanishes. The linear relationship between P and X_2 (Eq. (3)) then leads directly to the desired values P_{crit} that are associated with buckling.

The determinant of $\mathbf{S}(X_2)$ can be most conveniently obtained by expanding its fourth column. The determinant of $\mathbf{S}$ for $L_D = 2.0$ is plotted in Fig. 2. This graph shows the distribution of the values of X_2 (called roots) for which the determinant is singular. The oscillatory behavior about the axis induces that infinitely many roots exist.

For $L_D = 2.0$ the roots are all negative; this correlates with positive values for P. (The smallest critical axial force P_{crit} that causes buckling is associated with the negative root X_2 situated closest to the origin.) Hence, such a configuration requires compressive forces to cause buckling. This is not the case when $L_D > 2.64806$. For these configurations one or more roots are positive, which indicates that the inclined beam-column will buckle even if submitted to a limited traction. In these cases, the inclined beam-column buckles under its own weight. As an example, we exhibit the situation for $L_D = 2.8$ in Fig. 3; the first four roots are shown, one of which is positive.

[2] Using 15-digits float-point arithmetic, series expansions are limited to $-15 < X < 5$.

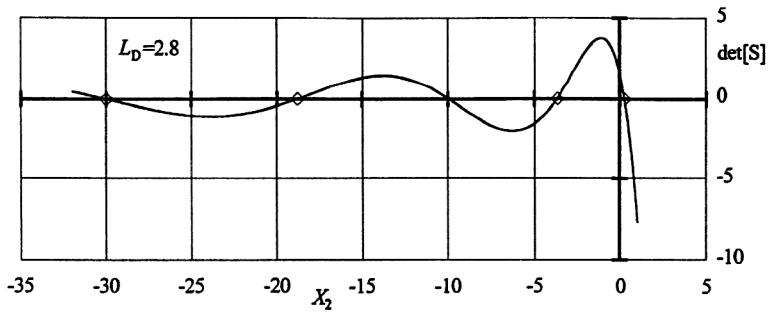

Fig. 3. Roots of the boundary determinant for $L_D = 2.8$.

Table 1
Roots of det[$\mathbf{S}(X_2)$]

L_D	First root	Second root	Third root
0.6	−27.115	−109.363	−246.435
0.7	−19.792	−80.219	−180.930
0.8	−15.021	−61.286	−138.390
0.9	−11.734	−48.289	−109.212
1.0	−9.368	−38.972	−88.327
1.1	−7.604	−32.078	−72.861
1.2	−6.251	−26.817	−61.086
1.3	−5.185	−22.712	−51.911
1.4	−4.329	−19.445	−44.621
1.5	−3.629	−16.800	−38.730
1.6	−3.044	−14.627	−33.900
1.7	−2.551	−12.817	−29.888
1.8	−2.129	−11.293	−26.519
1.9	−1.762	−9.996	−23.659
2.0	−1.440	−8.883	−21.211
2.1	−1.155	−7.918	−19.098
2.2	−0.900	−7.076	−17.259
2.3	−0.669	−6.335	−15.649
2.4	−0.458	−5.680	−14.231
2.5	−0.263	−5.098	−12.973
2.6	−0.082	−4.576	−11.853
2.7	0.086	−4.107	−10.850
2.8	0.244	−3.683	−9.947
2.9	0.295	−3.300	−9.132
3.0	0.539	−2.948	−8.392

Table 1 presents roots of det[$\mathbf{S}(X_2)$] for a large range of L_D. For small values of L_D, the roots have large absolute values, indicating that the critical forces are very high. Thus, depending upon the material of the inclined beam-column, it may crush by compression even before buckling occurs. There are also cases in which the first roots are positive, representing buckling under self-weight, as described above.

We next examine the behavior of the solution close to the critical points.

7. Behavior of the solutions in the neighborhood of a critical point

We will explore the behavior of a specific inclined beam-column configuration under different loading, giving particular attention to the behavior near critical values of the axial force, P_{crit}. Evaluating the solutions and plotting the center line will facilitate this study.

The dimensions of the beam-column to be analyzed are those for a regular API 5-inch drillpipe:

E	30.0×10^6 psi
OD	5 in
ID	4.276 in
w	19.5 lbf/ft
I	6.881×10^{-4} ft^4

The angle used is $\theta = 31°$ and the length is $L = 90$ ft. Using these numbers, the values of L_D and α are

L_D	1.60
α	33.790 ft

The angle of 31° was chosen because the resulting value for L_D (1.60) is in Table 1. (If the angle is selected as 45°, the value is $L_D = 1.50$.) In addition, the compressive axial force is given by

$$P = -940 \cdot X_2 \text{ lbf}.$$

We have shown that the generalized beam-column problem has a finite equilibrium solution for any value of the axial force P except for an infinite number of discrete values for which no solution exists. (These are the values of P that correspond to the values of X_2 representing the roots of $\det[\mathbf{S}(X_2)] = 0$.) For a given axial force P, the equilibrium configuration is found by solving the system in Eq. (5) for the coefficient vector $\mathbf{b}$. With $\mathbf{b}$, the solution $y(X)$ and its first and second derivatives are determined. Using physical arguments, one can show that equilibrium solutions for axial forces less than the first critical force are stable solutions; those for axial forces greater than the first critical force are unstable solutions.

The validity of the model is limited by the assumption of small displacements, which restricts how close the force can be to the critical values. In addition, the compressive limit of the beam-column places restrictions on how intense an axial force can be.

Table 1 shows that the first critical value of X_2 for this inclined beam-column example is −3.044. Fig. 4 displays a set of solution curves, each corresponding to a different value of X_2. On each of these graphs, the displacement $y(X)$ is plotted for $X_1 \leqslant X \leqslant X_2$, where $X_2 - X_1 = L_D = 1.6$. As X_2 decreases, X_1 decreases by the same amount, causing the graphs to

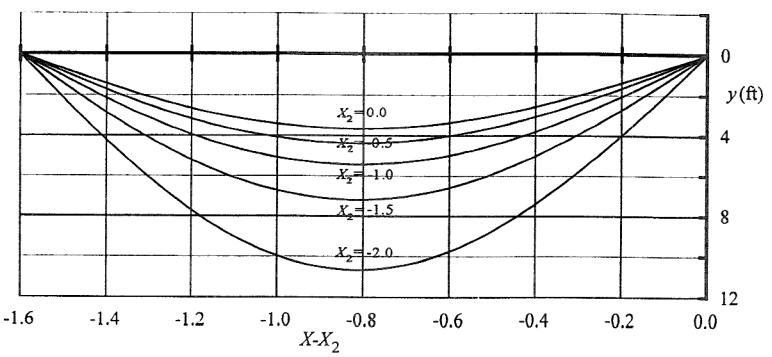

Fig. 4. Configuration for $P \to P_1^-$.

shift along the X-axis. The displacements along the column increase as the value of X_2 numerically decreases (increasing P) and approaches the critical value of -3.044; concomitantly, the axial compressive force P increases, approaching the first critical value $P_1 = 2861$ lbf. For ease in superposing these solution curves on the same graph, we have thus chosen the abscissa to be $X - X_2$ rather than X.

For $X_2 = 0$ ($P = 0$), the solution represents the condition in which the inclined beam-column is supported at its ends and bends due to self-weight. The solutions displayed on the graph are stable in the sense that if the inclined beam-column is perturbed from its equilibrium state, it will return to equilibrium when the perturbation is removed. In addition, if the axial force changes, the displacements adjust to the equilibrium.

Fig. 4 also shows the catastrophic behavior when the axial force approaches the critical value. It should be considered that, long before the critical force is reached, the system collapses and the model is no longer valid because the assumption of small displacement is violated. The behavior shown in Fig. 4 is called *subcritical buckling of first mode.*

We present yet another graph that describes the behavior of the beam-column as the axial force approaches the first critical value. A plot of the maximum displacement $y_{\max}$ versus X_2 is shown in Fig. 5 where the singularity can be observed for X_2 approaching the first critical value from the right ($X_2 \to -3.044^+, P \to P_1^-$).

8. "Virtual solutions": Axial force exceeding the first critical value

As a practical matter, the axial force cannot be increased to exceed the first critical force without collapsing the beam-column; however, it is theoretically possible to have equilibrium configurations for axial forces larger than the first critical force. These may be considered as "virtual solutions" that have intrinsic mathematical interest; they correspond to the shapes the center line must have in order to keep the beam-column in equilibrium. The equilibrium configurations for axial forces larger than the first critical force are unstable.

For X_2 approaching the first root from the left ($X_2 \to -3.044^-, P \to P_1^+$), the equilibrium configurations are those shown in Fig. 6. Due to the shape of the center line and the fact that the force is above the critical value, this behavior is referred to as *supercritical buckling of the first mode.* The plot of the maximum lateral displacement $y_{\max}$ versus X_2 is shown in Fig. 7.

Figs. 8 and 9 show the center line of the beam-column for several values of X_2 as the axial force gets close to the second critical value $X_2 = -14.627$. The limit between the first mode and the second mode occurs at $X_2 = -14.257$ when the derivative at the upper end of the column vanishes.

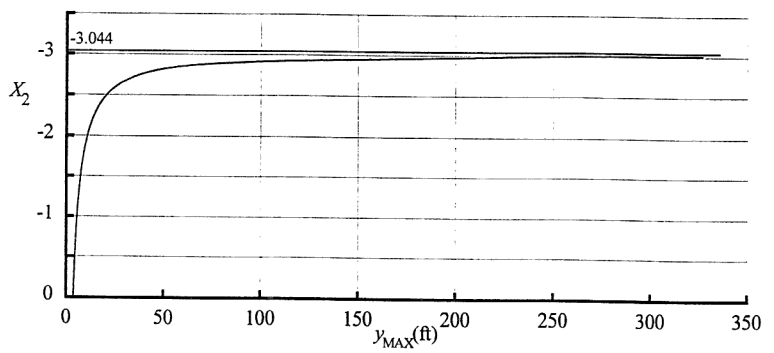

Fig. 5. Maximum lateral displacement for $P \to P_1^-$

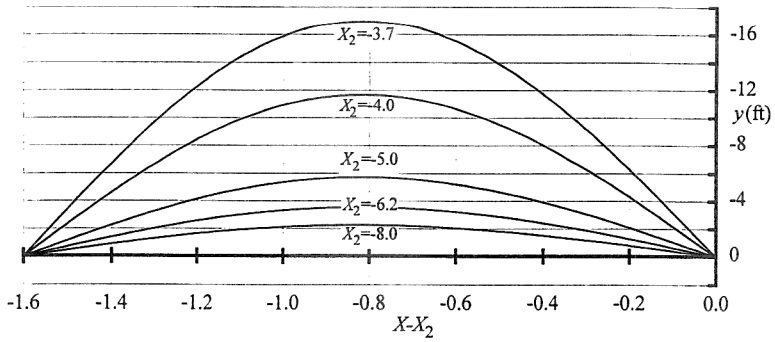

Fig. 6. Configuration for $P \rightarrow P_1^+$.

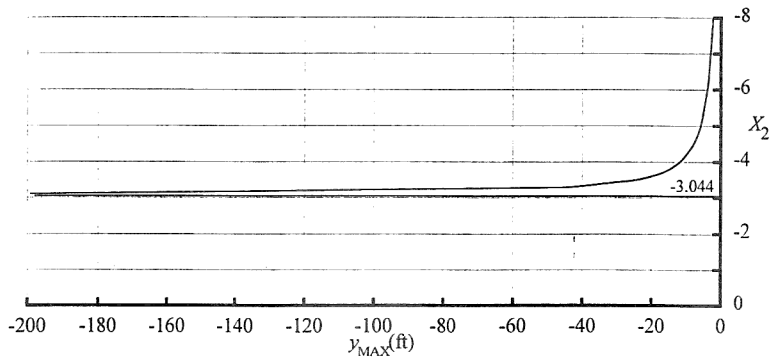

Fig. 7. Maximum lateral displacement for $P \rightarrow P_1^+$.

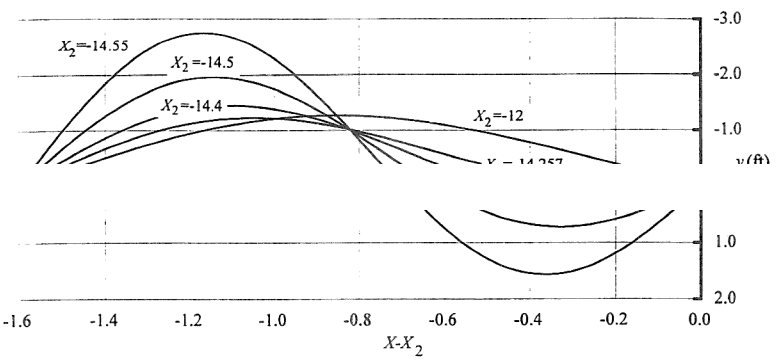

Fig. 8. Configuration for $P \rightarrow P_2^+$.

The region between the lower limit of the second mode and the second critical value is called the subcritical buckling of second mode. Fig. 9 also shows that the second mode ends at $X_2 = -14.985$ when the system enters into the region of *pseudo-critical buckling of first mode.*

Figs. 10 and 11 present the curves of the behavior about the third critical value at $X_2 = -33.900$. This region, in particular, is used to model the buckling of drillstrings within curved boreholes [5].

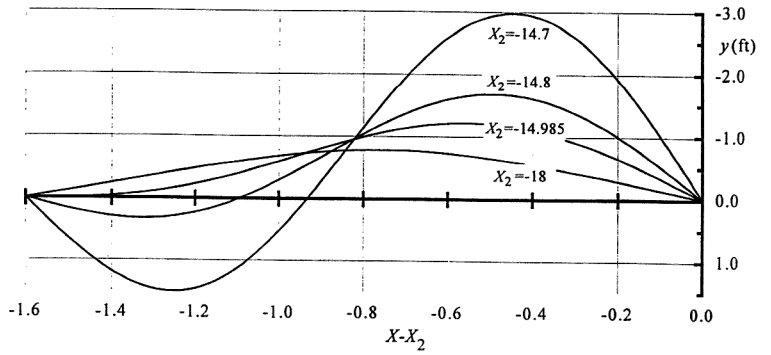

Fig. 9. Configuration for $P \rightarrow P_2^+$.

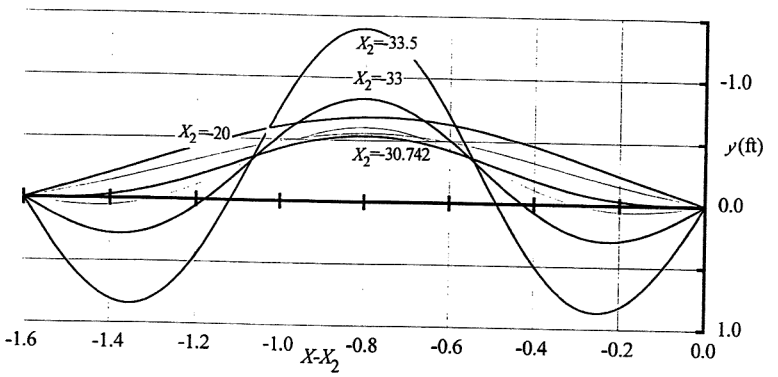

Fig. 10. Configuration for $P \rightarrow P_3^-$.

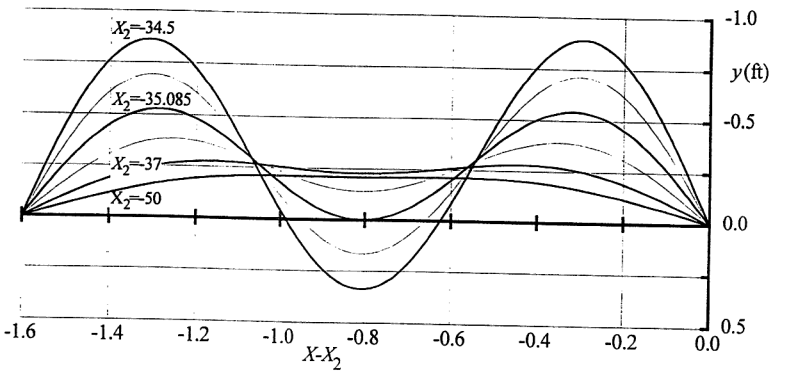

Fig. 11. Configuration for $P \rightarrow P_3^-$

9. Conclusions

The buckling behavior of an inclined beam-column is properly described by Eqs. (2) and (4). The expected singular behavior is demonstrated via the existence of singular points in the matrix of the non-homogeneous boundary system. The differential equation was obtained using the en-

ergy method and incorporates, as extreme cases, the differential equations for the flagpole and the beam-column. The analytical solution was obtained by series expansion and finally expressing the various series in closed form using generalized hypergeometric functions. The analytical solutions agree with the expected behavior of an inclined beam-column, especially the subcritical buckling of first mode, which is the only stable one. The study of higher modes has intrinsic mathematical interest, and, in addition, plays an important role in the modeling of the buckling of drillstrings within curved boreholes, as described by Sampaio [5].

Nomenclature

α	inclined beam-column constant
δ	axial displacement
θ	inclination (with respect to vertical)
EI	cross-section stiffness
${}_pF_q$	generalized hypergeometric functions
L	length
L_D	dimensionless length
P	compressive axial force
$\mathbf{S}$	boundary matrix
U	total potential energy
U_w	potential energy of external and body forces
V	strain energy of the inclined beam-column
w	linear distributed weight
W	work done by external and body forces
x	position along the inclined beam-column
X	dimensionless position
y	lateral displacement function

Appendix A

The derivation of the differential equation that describes an inclined beam-column is accomplished by applying the "energy method". This method, based upon the principle of virtual work, is the most suitable method for dealing with stability problems of structures [6].

In the context of our problem, this principle involves consideration of the decrease (or increase) of the potential energy of the forces acting upon the system (external forces and internal body forces) and also the increase (or decrease) of the strain energy of the beam-column.

Three assumptions are made: (1) the displacements are small; (2) the total effect of the distributed weight can be decomposed into two separate additive effects, one due to an axial distributed weight and the other due to a lateral distributed weight; and (3) the compressive part of the strain energy occurs in the early part of the loading process so that when the loading approaches the first critical force, the increase of the strain energy is due to the increase in bending. (This third assumption is admissible considering that in a buckling process, the bending increases much faster than the compression for an increase in the axial force. In addition, it implies that the energy due to shear stresses does not change significantly during the buckling process.)

To calculate the potential of external and body forces of the inclined beam-column, the system of forces acting upon the structure exhibited in Fig. 1 is decomposed as depicted schematically in

Fig. 12. The lateral forces at the supports cannot produce or absorb work since they cannot displace along their lines of action. The *total potential energy* of the system is given by

$$U = U_w + V,$$

where U represents the total potential energy of the system, U_w is the potential energy of the external and body forces, and V is the strain energy of the column.

The potential energy of external and body forces, like any potential, depends on a referential which can be chosen at will. We can then write:

$$U_w = U_0 - W.$$

W is the work done by external and body forces, since a force loses potential when it produces work and increases the potential when it absorbs work. The expression of the total potential energy becomes:

$$U = U_0 - W + V.$$

If a system in equilibrium is given a small displacement or deformation, the total potential energy does not change. Moreover, if the potential energy is both stationary and at a local minimum/maximum, the equilibrium is stable/unstable. Since we will be interested in the change of the potential, we can choose $U_0 = 0$ at the beginning of the buckling process. Therefore, the expression of the total potential energy will be given by

$$U = -W + V = -(W_P + W_{w_s} + W_{w_c}) + V. \tag{A.1}$$

The work W_P done by the axial force P (assumed essentially constant during the buckling process) is

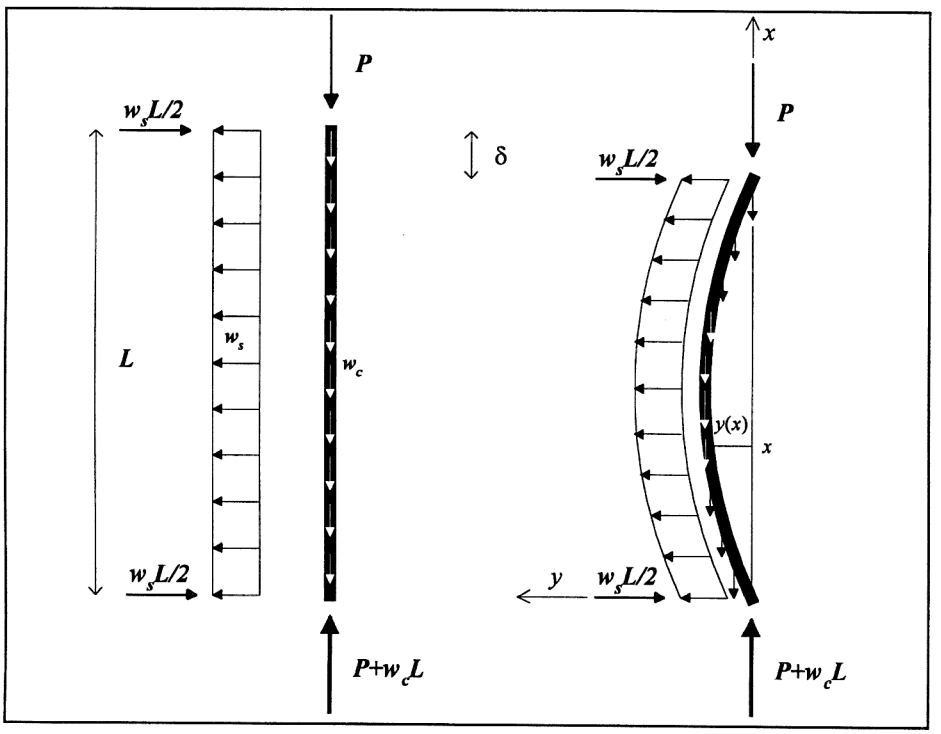

Fig. 12. Decomposition of the loading acting upon a beam-column.

$$W_P = P\delta.$$

The displacement δ (presumed small) may be approximated by

$$\delta \cong \int_0^L \left[\sqrt{1+(y')^2} - 1\right] \mathrm{d}x \cong \int_0^L \frac{(y')^2}{2}\,\mathrm{d}x,$$

where the square root term is approximated from the series expansion

$$\sqrt{1+(y')^2} = 1 + \frac{(y')^2}{2} - \frac{(y')^4}{8} + \frac{(y')^8}{16} - \frac{5(y')^{10}}{128} + \cdots \cong 1 + \frac{(y')^2}{2}.$$

Thus the work done by the external force P is given by

$$W_P = P\delta = \int_0^L P\frac{(y')^2}{2}\,\mathrm{d}x. \tag{A.2}$$

The work W_{w_s} done by the lateral component of the distributed weight w_s is given by

$$W_{w_s} = \int_0^L w_s y\,\mathrm{d}x, \tag{A.3}$$

and the work W_{w_c} done by the axial component of the distributed weight w_c is given by

$$W_{w_c} = \int_0^L w_c(L-x)\frac{(y')^2}{2}\,\mathrm{d}x, \tag{A.4}$$

where

$$w_s = w\,\sin(\theta),$$

$$w_c = w\,\cos(\theta).$$

The strain energy V is solely comprised of the bending term V_b, which is given by

$$V = V_b = \int_0^L \frac{EI}{2}(y'')^2\,\mathrm{d}x. \tag{A.5}$$

The expression for the total potential energy is obtained by inserting the expressions (A.2) through (A.5) into (A.1), resulting in

$$U(y) = \int_0^L \left\{\frac{EI}{2}(y'')^2 - \frac{1}{2}[P + w_c(L-x)](y')^2 - w_s y\right\}\mathrm{d}x. \tag{A.6}$$

This expression, which depends upon the displacement function $y(x)$, may be used to calculate the total potential energy for any configuration that the column may have. A direct application of the calculus of variations [7] to this functional leads to the determination of the function $y(x)$ for which the total potential energy becomes stationary. The functional can be represented by

$$U(y) = \int_0^L F(x,y,y',y'')\,\mathrm{d}x,$$

with

$$F(x,y,y',y'') = \frac{EI}{2}(y'')^2 - \frac{1}{2}[P + w_c(L-x)](y')^2 - w_s y.$$

The Eulerian differential equation and the set of boundary terms (conditions) are given by

$$\frac{d^2}{dx^2}\frac{\partial F}{\partial y''} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial y} = 0, \tag{A.7a}$$

$$\begin{cases} y, \dfrac{\partial F}{\partial y'} - \dfrac{d}{dx}\dfrac{\partial F}{\partial y''}, \\ y', \dfrac{\partial F}{\partial y''}. \end{cases} \tag{A.7b}$$

For a pinned-ends problem, the displacements at the ends (which are prescribed) are the essential boundary conditions, viz.,

$$y(0) = y(L) = 0.$$

Then the second pair of B.C. above are the natural ones, given by

$$\left.\frac{\partial F}{\partial y''}\right|_{x=0} = \left.\frac{\partial F}{\partial y''}\right|_{x=L} = 0. \tag{A.8}$$

Applying the prescription of (A.7a) and (A.8), we arrive at the differential equation and B.C.

$$\frac{d^2}{dx^2}[EIy''] + \frac{d}{dx}\{[P + w_c(L-x)]y'\} - w_s = 0,$$

$$y(0) = y(L) = y''(0) = y''(L) = 0.$$

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